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New Exact Solitary Wave Solutions of the KS Equation

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Two methods are described for obtaining new exact solitary wave solutions of the KS equation. Because these two methods are essentially equivalent the results obtained here are the same.

1. INTRODUCTION

Solving for the exact solitary wave solutions of nonlinear evolution equations has long been a major concern for both mathematicians and physicists. Although various methods for obtaining solitary wave solutions to nonlinear evolution equations have been established, it is not easy to find the exact solutions of some nonlinear evolution equations, particalarly nonintegrable nonlinear evolution equations. In this paper, by using two different methods, we study the exact solitary wave solutions of the Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxxx} = 0 \tag{1}$$

where α and β are constants. Equation (1) is a nonintegrable system, and it is one of the simplest nonlinear partial differential equations that exhibit chaotic behavior. Equation (1) was derived by Kuramoto and Tsuzaki (1975a, b) in the context of reaction–diffusion systems and by Sivashinsky (1977) for flame front propagation. Equation (1) has also appeared in other physical systems, such as long waves on thin films (Lin, 1974; Nakaya, 1975; Babchin, 1983; Topper and Kawahara, 1978), dissipative ion modes in plasmas (Laquey *et al.*, 1975; Cohen *et al.*, 1976), and interfacial instability between viscous

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fluids (Hooper and Grimshaw, 1985). Here we look for the traveling solution of Eq. (1), that is, a solution on the form

$$u(x, t) = u(\xi), \qquad \xi = k(x - wt) \tag{2}$$

where k, w are constants to be determined.

Substituting (2) into (1), we have

$$-wu + \frac{1}{2}u^2 + aku_\xi + \beta k^3 u_{\xi\xi\xi} = C \tag{3}$$

where C is an integration constant.

2. SPECIAL TRUNCATED EXPANSION METHOD

The special truncated expansion method was introduced by Lu *et al.* (1993). They assume the following special truncated expansion:

$$u(x, t) = u(\xi) = \sum_{n=0}^{N} A_n F^n \tag{4}$$

where

$$F = \frac{1}{1 + e^{\xi}}, \qquad \xi = k(x - wt) \tag{5}$$

From (4), we have

$$\frac{d^k u}{d\,s^k} = \sum_{n=1}^{N} A_n \sum_{n=0}^{k} A_{nm}^{(k)} F^{n+m} \tag{6}$$

$$A_{nm}^{(k+1)} = (n + m - 1)A_{n(m-1)}^{(k)} - (n + m)A_{nm}^{(k)} \tag{7}$$

$$A_{nm}^{(k)} = (-1)n(n + 1)\cdots(n + m - 1) \sum_{\substack{j1+\cdots+j_m k-m \\ 0<j_1,\cdots,j_m \le k-m}} n^{j_1} \cdots (n + m)^{j_n} \tag{8}$$

$$A_{nn}^{0} = 1, \qquad n \ge 1 \tag{9}$$

We balance the highest power of F in the nonlinear term uu_x with the highest power of F in the fourth-order derived term u_{xxxx} in Eq. (1) to obtain $2N + 1 = N + 4$, so that $N = 3$. Thus we let

$$u(x, t) = u(\xi) = A_0 + A_1 F + A_2 F^2 + A_3 F^3 \tag{10}$$

Substituting (10) into (2) and collecting term with the same power of F, we have

$$F^6: \quad \frac{1}{2}A_3^2 + 60\beta k^3 A_3 = 0 \tag{11}$$

$$F^5: \quad A_2A_3 + \beta k^3(24A_2 - 144A_3) = 0 \tag{12}$$

$$F^4: \quad \frac{1}{2}(A_2^2 + 2A_1A_3) + 3\alpha k A_3 + \beta k^3(6A_1 - 54A_2 + 111A_3) = 0 \tag{13}$$

$$F^3: \quad -w A_3 + (A_0 A_3 + A_1A_2) + \alpha k(2A_2 - 3A_3) + \beta k^3(-12A_1 + 38A_2 - 27A_3) = 0 \tag{14}$$

$$F^2: \quad -w A_2 - \frac{1}{2}(A_1^2 + 2A_0 A_2) + \alpha k(A_1 - 2A_2) + \beta k^3(7A_1 - 8A_2) = 0 \tag{15}$$

$$F^1: \quad A_1(-w + A_0 - \alpha k - \beta k^3) = 0 \tag{16}$$

$$F^0: \quad -w A_0 + \frac{1}{2}A_0^2 = C \tag{17}$$

From (11), we obtain

$$A_3 = -120\beta k^3 \tag{18}$$

Then substituting (18) into (12), we find

$$A_2 = 180\beta k^3 \tag{19}$$

From (16), we can give two cases: (i) For

$$A_1 = 0 \tag{20}$$

from (13)–(15), we find

$$k^2 = -\frac{\alpha}{19\beta}, \qquad w = 30\beta\, k^3 + A_0 \tag{21}$$

(ii) For $A_1 \neq 0$, from (13)–(15), we find

$$k^2 = \frac{-11\alpha}{19\beta}, \qquad w = A_0 - \alpha k - \beta k^3 \tag{22}$$

$$A_1 = -62\beta k^3 - 2\alpha k \tag{23}$$

So we have two new kinds of solitary wave solutions of Eq. (1):

$$u_1(x, t) = u_1(\xi) = A_0 + 180\beta k_1^3 F^2 - 120\beta k_1^3 F^3 \tag{24}$$

where $k_1 = (-\alpha/19\beta)^{1/2}$, $w_1 = 30\beta k_1^3 + A_0$; and

$$u_2(x, t) = u_2(\xi) = A_0 - \frac{720\alpha k}{19} F + 180\beta k^3 F^2 - 120\beta k^3 F^3 \tag{25}$$

where $k_2 = (11\alpha/19\beta)^{1/2}$, $w_2 = A_0 - \alpha k_2 - \beta k_2^3$.

Noting that

$$F(\xi) = \frac{1 - \tanh(\xi/2)}{2} \tag{26}$$

we find that solutions (24) and (25) become

$$u_1(x, t) = w_1 + \frac{90}{19} \alpha k_1 \tanh[k_1(x - w_1 t)] + 120\beta k^3 \tanh^3[k_1(x - w_1 t)] \tag{27}$$

where $k_1 = (-\alpha/76\beta)^{1/2}$, $w_1 = -(60/19)\alpha k_1 + A_0$; and

$$u_2(x, t) = w_2 - \frac{270}{19} \alpha k_2 \tanh[k_2 (x - w_2 t)] + 120\beta k_2^3 \tanh^3[k_2(x - w_2 t)] \tag{28}$$

where $k_2 = (11\alpha/76\beta)^{1/2}$, $w_2 = -(60/19)\, \alpha k_2 + A_0$

3. HYPERBOLIC TANGENT FUNCTION METHOD

The hyperbolic tangent function method was has been used extensively (Heremann *et al.*, 1986; Lan and Wang, 1989; Malfliet, 1992). They assume the following hyperbolic tangent function expansion:

$$u(x, t) = u\,(\xi) = \sum_{n=0}^{N} a_n \tanh \xi^n \tag{29}$$

We balance the highest power of tanh ξ in the nonlinear term uu_x with the highest power of tanh ξ in the fourth-order derived term u_{zzzz} in Eq. (1) to obtain $2N + 1 = N + 4$, so that $N = 3$. Thus we let

$$u(x, t) = u(\xi) = a_0 + a_1 \tanh \xi + a_2 \tanh^2 \xi + a_3 \tanh^3 \xi \tag{30}$$

Substituting (30) into (2) and collecting term with same power of tanh ξ, we have

$$\tanh^6 \xi: \qquad \frac{1}{2} a_3^2 - 60 a_3 \beta k^3 = 0 \tag{31}$$

$$\tanh^5 \xi: \qquad a_2\,(a_3 + 24\beta k^3) = 0 \tag{32}$$

$$\tanh^4 \xi: \quad \frac{1}{2} a_2^2 + a_1 a_3 - 3\alpha k a_3 - 6\beta k^3 a_1 + 114\beta k^3 a_3 = 0 \tag{33}$$

$$\tanh^3 \xi: \quad - w a_3 + a_1 a_2 + a_0 a_3 - 2\alpha k a_2 - 40\beta k^3 a_2 = 0 \tag{34}$$

$$\tanh^2 \xi: \quad -w a_2 + \frac{1}{2} a_1^2 + a_0 a_2 - \alpha k a_1 + 3\alpha k a_3 + 8\beta k^3 a_1 - 60\beta k^3 a_3 = 0 \tag{35}$$

$$\tanh^1 \xi: \quad -w a_1 + a_0 a_1 - 2\alpha k a_2 + 16\beta k^3 a_2 = 0 \tag{36}$$

$$\tanh^0 \xi: \quad -w a_0 + \frac{1}{2} a_0^2 + \alpha k a_1 + 6\beta k^3 a_3 = 0 \tag{37}$$

From (31), we obtain

$$a_3 = -120\beta k^3 \tag{38}$$

From (32), we have

$$a_2 = 0 \tag{39}$$

From (34), we have

$$a_0 = w \tag{40}$$

From (33), we find

$$a_3 = 120\beta k^3 + \frac{60}{19}\alpha k \tag{41}$$

From (35), we obtain

$$(-\alpha - 76\beta k^2)(11\alpha - 76\beta k^2) = 0 \tag{42}$$

so k has two solutions

$$k_1^2 = \frac{-\alpha}{76\beta}, \qquad k_2^2 = \frac{11\alpha}{76\beta} \tag{43}$$

Considering (38)–(43), we obtain two new kinds of solitary wave solutions of Eq. (1),

$$u_1(x, t) = w + \frac{90}{19}\alpha k_1 \tanh[k_1(x - wt)] + 120\beta k_1^3 \tanh^3[k_1(x - wt)] \tag{44}$$

$$u_2(x, t) = w - \frac{270}{19}\alpha k_2 \tanh[k_2(x - wt)] + 120\beta k_2^2 \tanh^3[k_2(x - wt)] \tag{45}$$

where $k_1 = (-\alpha/76\beta)^{1/2}$ or $k_2 = (11\alpha/76\beta)^{1/2}$, and w is an arbitrary constant.

4. CONCLUSION

In summary, we have obtained two kinds of solitary wave solutions of the KS equation by two different methods. In fact, these two methods are essentially equivalent because we have the expression $F(\xi) = 1/2\,[1 - \tanh(\xi/2)]$, so we obtain the same solitary wave solutions of Eq. (1). It would be interesting to study other nonlinear evolution equations by using these two different methods; we leave this to future study.

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